Integr. Equ. Oper. Theory 77 (2013), 279–290
DOI 10.1007/s00020-013-2083-z
Published online August 4, 2013

Integral Equations and Operator Theory

Two Inner Sequences Based Invariant Subspaces in H^2 $(\mathbb{D}^2)$

Yixin Yang

Abstract. Let M be a shift invariant subspace in the vector-valued Hardy space $H^2_E(\mathbb{D})$. The Beurling–Lax–Halmos theorem says that M can be completely characterized by $\mathcal{B}(E)$-valued inner function Θ. When $E = H^2(\mathbb{D})$, $H^2_E(\mathbb{D})$ is the Hardy space on the bidisk $H^2(\mathbb{D}^2)$. Recently, Qin and Yang (Proc Am Math Soc, 2013) determines the operator valued inner function $\Theta(z)$ for two well-known invariant subspaces in $H^2(\mathbb{D}^2)$. This paper generalizes the $\Theta(z)$ by Qin and Yang (Proc Am Math Soc, 2013) and deal with the structure of $M = \Theta(z)H^2(\mathbb{D}^2)$ when M is an invariant subspace in $H^2(\mathbb{D}^2)$. Unitary equivalence, spectrum of the compression operator and core operator are studied in this paper.

Mathematics Subject Classification (2010). Primary 47A13; Secondary 46E20.

Keywords. Hardy space over the bidisc, invariant subspace, operator-valued inner function, inner sequence, unitary equivalence, spectrum, core operator.

1. Introduction

Let $\mathbb{D}$ be the unit disk with boundary $\mathbb{T}$ in the complex plane $\mathbb{C}$. Let E be a separable Hilbert space of infinite dimension and $\mathcal{B}(E)$ be the set of bounded linear operators on E. Let $H^2_E(\mathbb{D})$ be the E-valued Hardy space on the unit disk. A $\mathcal{B}(E)$-valued analytic function $\Theta(z)$ is said to be an inner function if $\|\Theta(z)\| \leq 1$ for each $z \in \mathbb{D}$, and $\Theta(z)$ is almost everywhere an isometry on $\mathbb{T}$. Let T_z be the shift operator on $H^2_E(\mathbb{D})$, i.e., $T_z f(z) = zf(z), f \in H^2_E(\mathbb{D})$. A closed subspace M of $H^2_E(\mathbb{D})$ is said to be invariant if $T_z M \subseteq M$. Beurling–Lax–Halmos theorem (cf. [2,5,9]) says that such M is of the form $M = \Theta(z)H^2_E(\mathbb{D})$ for some $\mathcal{B}(E)$-valued inner function Θ. When $E = H^2(\mathbb{D})$, $H^2_E(\mathbb{D})$ is then the Hardy space on the bidisk $\mathbb{D}^2$, denoted by $H^2(\mathbb{D}^2)$.

Let z and w be the coordinate functions on $H^2(\mathbb{D}^2)$ and T_z and T_w be the corresponding multiplication operators. It is well known that T_z and T_w are shifts of infinite multiplicity. The classical Hardy space in variable

z (or w) are denoted by $H^2(z)$ (or $H^2(w)$) respectively. M is said to be an invariant subspace if it is invariant both under the action of T_z and T_w. Let $N = H^2(\mathbb{D}^2) \ominus M$ be the corresponding quotient space. By the Beurling–Lax–Halmos theorem, the T_z-invariant subspace M is of the form $M = \Theta(z)H^2(\mathbb{D}^2)$ for some $\mathcal{B}(H^2(w))$-valued inner function Θ. So for an invariant subspace M, there is also a $\mathcal{B}(H^2(z))$-valued inner function Φ such that $M = \Phi(w)H^2(\mathbb{D}^2)$. It is clear that Θ and Φ are closely related to each other. All the information of M are encoded in the two operator-valued inner functions Θ and Φ. However, the structure of invariant subspaces in $H^2(\mathbb{D}^2)$ is far more complex. In general, it is very hard to determine these two functions. Recently, [11] determined these two functions for two well-known invariant subspaces [8,12,13] in $H^2(\mathbb{D}^2)$ and they turn out to be strikingly simple.

We will focus on the invariant subspace given by the following operator-valued inner function.

Definition 1.1. An infinite sequence of inner functions $\{\varphi_j(z)\}_{j\geq 0}$ in $H^2(z)$ is called inner sequence if it satisfies one of the following conditions:

(1) $\{\varphi_j(z)\}_{j\geq 0}$ is decreasing, that is, every φ_j/φ_{j+1} is a nonconstant inner function;
(2) $\{\varphi_j(z)\}_{j\geq 0}$ is increasing, that is, every φ_{j+1}/φ_j is a nonconstant inner function.

Definition 1.2. An infinite sequence of orthogonal projections $\{P_j\}_{j\geq 0}$ in $\mathcal{B}(H^2(w))$ is called orthogonal complementary projections if the range of P_j, denoted by $ranP_j$, are piecewise orthogonal to each other and $\sum_{j=0}^{\infty} P_j = I$ in the sense of strong operator topology.

It can be checked that

$$\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j \tag{1.1}$$

is an inner function. We can define a T_z-invariant subspace M in $H^2(\mathbb{D}^2)$ as follows:

$$M = \Theta(z)H^2(\mathbb{D}^2).$$

When M is an invariant subspace of $H^2(\mathbb{D}^2)$, it will be called two inner sequences based invariant subspace, which will be seen clearly in the following Theorem 2.2. It should be pointed out that this type of invariant subspace has been studied in [7] and rank of $H^2(\mathbb{D}^2) \ominus M$ for the pair T_z^* and T_w^* was determined.

This paper will study some properties of such type invariant subspace. In Sect. 2, the structure of such invariant subspace can be determined completely. In Sect. 3, we will discuss the unitary equivalence of two such invariant subspaces. In Sect. 4, we will determine the spectrum of the compression operators. In the Sect. 5, the trace and rank of the core operator will be calculated exactly.

2. Structure of M

In the following, without loss of generality, we always assume that the greatest common factor of an inner sequence is a constant, so when $\{\psi_j(w)\}_{j\geq 0}$ is an increasing inner sequence, $\psi_0(w) = 1$. The following lemma will be used frequently.

Lemma 2.1. *Let $\{\varphi_j(z)\}_{j\geq 0}$ be a decreasing inner sequence and $\{\psi_j(w)\}_{j\geq 0}$ be an increasing inner sequence. Then*

$$M = \sum_{j=0}^{\infty} \varphi_j(z) H^2(z) \otimes (\psi_j H^2(w) \ominus \psi_{j+1}(w) H^2(w)) \tag{2.1}$$

is an invariant subspace. Moreover, if setting $\varphi_{-1}(z) = 0$, we have

$$M \ominus zM = \sum_{j=0}^{\infty} \mathbb{C}\varphi_j(z) \otimes (\psi_j(w) H^2(w) \ominus \psi_{j+1}(w) H^2(w)); \tag{2.2}$$

and

$$M \ominus wM = \sum_{j=0}^{\infty} (\varphi_j(z) H^2(z) \ominus \varphi_{j-1}(z) H^2(z)) \otimes \mathbb{C}\psi_j(w). \tag{2.3}$$

Proof. It is easy to check that M is T_z-invariant and (2.2) holds. We need to show that M is also T_w-invariant.

Since $\{\varphi_j(z)\}_{j\geq 0}$ is a decreasing inner sequence, then

$$\varphi_0(z) H^2(z) \subset \varphi_1(z) H^2(z) \subset \cdots .$$

Note that $\varphi_{-1}(z) = 0$ and $\psi_0(w) = 1$, we have

$$\sum_{j=k}^{\infty} (\psi_j(w) H^2(w) \ominus \psi_{j+1}(w) H^2(w)) = \psi_k(w) H^2(w)$$

and

$$\sum_{j=0}^{k} (\varphi_j(z) H^2(z) \ominus \varphi_{j-1}(z) H^2(z)) = \varphi_k(z) H^2(z)$$

for any $k = 0, 1, \ldots$. Therefore we can rewrite M as the form

$$M = \sum_{j=0}^{\infty} (\varphi_j(z) H^2(z) \ominus \varphi_{j-1}(z) H^2(z)) \otimes \psi_j(w) H^2(w). \tag{2.4}$$

Then M is T_w-invariant and (2.3) also follows easily. $\square$

Let $\Theta(z)$ be an operator-valued inner function as in (1.1) and $M = \Theta(z) H^2(\mathbb{D}^2)$ be a T_z-invariant subspace in $H^2(\mathbb{D}^2)$. The following theorem shows that M is also T_w-invariant only when M is of the form (2.1). For simplicity, it is enough to consider the case that $\{\varphi_j(z)\}_{j\geq 0}$ is decreasing.

Theorem 2.2. *Let $\{\varphi_j(z)\}_{j\geq 0}$ be a decreasing inner sequence and $\{P_j\}_{j\geq 0}$ be orthogonal complementary projections in $\mathcal{B}(H^2(w))$. Let $\Theta(z)=\sum_{j=0}^{\infty}\varphi_j(z)P_j$ and $M = \Theta(z)H^2(\mathbb{D}^2)$. Then $M = \Theta(z)H^2(\mathbb{D}^2)$ is an invariant subspace if and only if there exists an increasing inner sequence $\{\psi_j(w)\}_{j\geq 0}$ with $ranP_j = \psi_j H^2(w) \ominus \psi_{j+1}(w)H^2(w)$ for $j = 0, 1, 2, \ldots$ such that M is of the form (2.1).*

Moreover, there are orthogonal complementary projections $\{Q_j\}_{j\geq 0}$ in $\mathcal{B}(H^2(z))$ such that $M = \Phi(w)H^2(\mathbb{D}^2)$, where $\Phi(w) = \sum_{j=0}^{\infty}\psi_j(w)Q_j$.

Proof. If M is of the form (2.1), by Lemma 2.1, M is an invariant subspace.

We need to show when $M = \Theta(z)H^2(\mathbb{D}^2)$ is an invariant subspace in $H^2(\mathbb{D}^2)$, then M is of the form (2.1). Since $\Theta(z) = \sum_{j=0}^{\infty}\varphi_j(z)P_j$, where $\{P_j\}_{j\geq 0}$ are the orthogonal complementary projections in $\mathcal{B}(H^2(w))$, we have

$$\begin{aligned} M &= \Theta(z)H^2(\mathbb{D}^2) \\ &= \sum_{j=0}^{\infty}\varphi_j(z)H^2(z)\otimes ranP_j. \end{aligned} \tag{2.5}$$

It is clear that

$$N = \sum_{j=0}^{\infty}(H^2(z)\ominus\varphi_j(z)H^2(z))\otimes ranP_j. \tag{2.6}$$

Since M is T_w-invariant, so for any $h(z)\in H^2(z)$, $g_j(w)\in ranP_j$, we have $\varphi_j(z)h(z)wg_j(w)\in M$, and this implies that

$$\langle\varphi_j(z)h(z)wg_j(w), h_i(z)f_i(w)\rangle = 0,$$

i.e.,

$$\langle\varphi_j(z)h(z), h_i(z)\rangle\langle wg_j(w), f_i(w)\rangle = 0 \tag{2.7}$$

for any $i\neq j$, $h_i(z)\in H^2(z)\ominus\varphi_i(z)H^2(z)$ and $f_i(w)\in ranP_i$, where $\langle\cdot,\cdot\rangle$ is the inner product in the corresponding spaces.

Since $\{\varphi_j(z)\}_{j\geq 0}$ is a decreasing inner sequence, then for $i<j$, there are $h(z)\in H^2(z)$ and $h_i(z)\in H^2(z)\ominus\varphi_i(z)H^2(z)$ such that $\langle\varphi_j(z)h(z), h_i(z)\rangle\neq 0$, otherwise, $\varphi_j(z)H^2(z)\subseteq\varphi_i(z)H^2(z)$, this contradicts with $\varphi_i(z)/\varphi_j(z)$ is a nonconstant inner function. Therefore, (2.7) implies that

$$\langle wg_j(w), f_i(w)\rangle = 0$$

for any $i<j$ and any $g_j(w)\in ranP_j$, $f_i(w)\in ranP_i$. Pick $j = 1, 2\ldots$, we get $w(ranP_j)\subseteq\sum_{k=j}^{\infty}ranP_k$. Thus

$$w\left(\sum_{k=j}^{\infty}ranP_k\right)\subseteq\sum_{k=j}^{\infty}ranP_k,$$

which means that $\sum_{k=j}^{\infty}ranP_k$ is an invariant subspace in $H^2(w)$. By Beurling theorem, there is a sequence of inner functions $\{\psi_j(w)\}_{j\geq 0}$ such that

$$\sum_{k=j}^{\infty}ranP_k = \psi_j(w)H^2(w).$$

Since $\sum_{k=0}^{\infty} ranP_k = H^2(w)$, so $\psi_0(w) = 1$. It is not hard to see that

$$\psi_j(w)H^2(w) \subseteq \psi_{j-1}(w)H^2(w),$$

hence $\{\psi_j(w)\}_{j\geq 0}$ is an increasing inner sequence. Note that

$$ranP_j = \psi_j H^2(w) \ominus \psi_{j+1}(w)H^2(w), j = 0, 1, \ldots.$$

So by (2.5), M is of the form

$$M = \sum_{j=0}^{\infty} \varphi_j(z)H^2(z) \otimes (\psi_j H^2(w) \ominus \psi_{j+1}(w)H^2(w)),$$

which is the form (2.1).

Let Q_j be the orthogonal projection from $H^2(z)$ onto $\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)$, $j = 0, 1, \ldots$. Since $\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)$ are orthogonal to each other and the greatest common factor of $\{\varphi_j(z)\}_{j\geq 0}$ is a constant inner function, we have $\sum_{j=0}^{\infty}(\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)) = H^2(z)$ and $\sum_{j=0}^{\infty} Q_j = I$ on $H^2(z)$. Now define the $\mathcal{B}(H^2(z))$-valued function

$$\Phi(w) = \sum_{j=0}^{\infty} \psi_j(w)Q_j.$$

For every $g \in H^2(z)$ and almost every $w \in \mathbb{T}$, we have

$$\begin{aligned}\|\Phi(w)g\|^2 &= \sum_{j=0}^{\infty} \|\psi_j(w)Q_j g\|^2\\ &= \sum_{j=0}^{\infty} \|Q_j g\|^2\\ &= \|g\|^2,\end{aligned}$$

which means Φ is inner. Since M is now of the form (2.1), by (2.3) we have $M \ominus wM = \Phi(w)H^2(z)$ and therefore $M = \Phi(w)H^2(\mathbb{D}^2)$. This completes the proof. □

Remark 2.3. When $\psi_j(w) = w^j$, $j = 0, 1, \ldots, M = \sum_{j=0}^{\infty} \varphi_j(z)H^2(z)w^j$ is called the inner based invariant subspaces by Seto and Yang in [13]. The inner based invariant subspaces have been very well studied in [11–13], and in particular, when $\varphi_0(z)$ is a Blaschke product, Izuchis (cf. [6]) determined the rank of M completely.

3. Unitary Equivalence

It is well known that in the one variable case, all invariant subspaces are unitarily equivalent to $H^2(z)$ by Beurling's theorem. However, there are non-unitarily equivalent invariant subspaces (cf. [3]). In [12], the unitarily equivalent for inner based invariant subspaces are described completely and they turn out to be simple. In this section, we will restrict the unitarily equivalent for two inner sequences based invariant subspaces and it is also simple in this situation.

Theorem 3.1. *Let* $M = \sum_{j=0}^{\infty} \varphi_j(z)H^2(z) \otimes (\psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w))$ *and* $\widetilde{M} = \sum_{j=0}^{\infty} \widetilde{\varphi_j(z)}H^2(z) \otimes (\widetilde{\psi_j(w)}H^2(w) \ominus \widetilde{\psi_{j+1}(w)}H^2(w))$*, where* $\{\varphi_j(z)\}_{j\geq 0}$ *is decreasing and* $\{\psi_j(z)\}_{j\geq 0}$ *is increasing and* $\psi_0 = 1, \widetilde{\psi_0} = 1$*. Then* M *and* $\widetilde{M}$ *are unitarily equivalent if and only if there exists a unimodular function* $q = q(z)$ *depending only on* z *such that* $M = q(z)\widetilde{M}$*.*

Proof. If M and $\widetilde{M}$ are unitarily equivalent, by the theorem in [1], there exists a unimodular function q such that $M = q\widetilde{M}$. Since $q(z,w)\widetilde{\varphi_0(z)}$ and $\overline{q(z,w)}\varphi_0(z)$ are in $H^2(\mathbb{D}^2)$, $q(z,w)$ is w-analytic and conjugate w-analytic. Hence q depends only on z. The converse is trivial. □

4. Spectrum

Let $H^2(\mathbb{D})$ be the classical Hardy space on the unit disk. For an inner function θ and an invariant subspace $\theta H^2(\mathbb{D})$ the Jordan block $S(\theta)$ is defined on the quotient space $N_\theta = H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$ by

$$S(\theta)f = P_\theta z f, \quad f \in N_\theta,$$

where P_θ is the orthogonal projection from $H^2(\mathbb{D})$ onto N_θ.

For an invariant subspace M of $H^2(\mathbb{D}^2)$, let $N = H^2(\mathbb{D}^2) \ominus M$. The compression of T_z and T_w on N are denoted by S_z and S_w respectively, that is,

$$S_z = P_N T_z|_N, \quad S_w = P_N T_w|_N.$$

where P_N is the orthogonal projection from $H^2(\mathbb{D}^2)$ onto N.

For the two inner based sequences invariant subspace defined in Lemma 2.1, S_z and S_w have close relationship with the Jordan block and we will describe it now.

By (2.1) and (2.4), we have

$$\begin{aligned} N &= \sum_{j=0}^{\infty} (H^2(z) \ominus \varphi_j H^2(z)) \otimes (\psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w)) \\ &= \sum_{j=0}^{\infty} (\varphi_{j+1}(z)H^2(z) \ominus \varphi_j(z)H^2(z)) \otimes (H^2(w) \ominus \psi_{j+1}(w)H^2(w)). \end{aligned} \tag{4.1}$$

Let $f = \sum_{j=0}^{\infty} f_j(z)g_j(w) \in N$, where $f_j \in H^2(z) \ominus \varphi_j H^2(z)$, $g_j \in \psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w)$, we have

$$\begin{aligned} S_z f &= \bigoplus_{j=0}^{\infty} P_{\varphi_j}\left(\sum_{j=0}^{\infty} z f_j(z) g_j(w)\right) = \sum_{j=0}^{\infty}\left(\bigoplus_{j=0}^{\infty} P_{\varphi_j} z f_j(z) g_j(w)\right) \\ &= \sum_{j=0}^{\infty} (S(\varphi_j) f_j) g_j(w). \end{aligned}$$

So with respect to the decomposition (4.1), S_z is the Jordan operator (cf. [10])

$$S_z = \sum_{j=0}^{\infty} (\underbrace{S(\varphi_j) \oplus \cdots \oplus S(\varphi_j)}_{d_j - times}), \tag{4.2}$$

where d_j is the dimension of $\psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w)$. Using the similar argument, we can also get the decomposition for S_w,

$$S_w = \sum_{j=1}^{\infty} (\underbrace{S(\psi_j) \oplus \cdots \oplus S(\psi_j)}_{l_j - times}), \tag{4.3}$$

where l_j is the dimension of $\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)$.

We will determine the spectrum of S_z and S_w. From Theorem 2.2 and the proof, $M \ominus wM = \Phi(w)H^2(w)$, where $\Phi(w) = \sum_{j=0}^{\infty} \psi_j(w)Q_j$ and Q_j is the orthogonal projection from $H^2(z)$ onto $\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)$, $j = 0, 1, 2, \ldots$. By the model theory of Nagy and Foias (cf. [10]), $\sigma(S_w) = \sigma(\Phi)$, where $\sigma(\Phi)$ consists all $\lambda \in \mathbb{D}$ such that $\Phi(\lambda)$ is not invertible, and all $\xi \in \mathbb{T}$ such that Φ can not be extended analytically to a neighborhood U of ξ and Φ is unitary-valued on $U \cap \mathbb{T}$.

Remark 4.1. If $\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j$ is a nontrivial infinite sum, i.e., all P_j are nonzero projection from $H^2(w)$ onto $\psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w)$, $j = 0, 1, \ldots$. Then ψ_{j+1}/ψ_j is a nontrivial inner function and $\sum_{j=0}^{\infty} Q_j = I$, so $\lim_{j\to\infty} \psi_j(\lambda) = 0$ for every $\lambda \in \mathbb{D}$.

Remark 4.2. If $\Theta(z)$ is a finite sum, i.e., there is N such that $P_j \neq 0$ for $j < N$, $P_j = 0$ for $j \geq N$. Then we have $\psi_j = \psi_N$ for all $j \geq N$.

Theorem 4.3. *Let M be the two inner sequences based invariant subspace defined in Lemma 2.1, then $\sigma(S_z) = \sigma(\varphi_0)$. If $\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j$ is a nontrivial infinite sum, i.e., $\{\psi_j(w)\}_{j\geq 0}$ is a nontrivial infinite inner sequence, $\sigma(S_w) = \overline{\mathbb{D}}$; if $\Theta(z)$ is a finite sum, i.e., $\psi_j = \psi_N$ for all $j \geq N$, $\sigma(S_w) = \sigma(\psi_N)$.*

Proof. Since $\{\varphi_j(z)\}_{j\geq 0}$ is a decreasing inner sequence, we have $\sigma(\varphi_0) \supseteq \sigma(\varphi_1) \supseteq \cdots$ and so $\sigma(S(\varphi_0)) \supseteq \sigma(S(\varphi_1)) \supseteq \cdots$. (4.2) implies that $\sigma(S_z) = \bigcup_{j=0}^{\infty} \sigma(S(\varphi_j)) = \sigma(S(\varphi_0)) = \sigma(\varphi_0)$.

Now suppose that $\Theta(z) = \sum_{j=0}^{\infty} \varphi_j(z)P_j$ is a nontrivial infinite sum. By the definition of inner sequence, φ_j/φ_{j+1} is a nonconstant inner function, so each Q_j is nonzero projection. Choose unit vectors $\{f_j\}_{j=0}^{\infty}$ with $f_j \in ranQ_j$, then for every $\lambda \in \mathbb{D}$, $\lim_{j\to\infty} \|\Phi(\lambda)f_j\| = \lim_{j\to\infty} |\psi_j(\lambda)| = 0$, which means that $\Phi(\lambda)$ is not invertible. So $\lambda \in \sigma(S_w)$ and thus $\mathbb{D} \subset \sigma(S_w)$. Note that S_w is a contraction, we obtain $\sigma(S_w) = \overline{\mathbb{D}}$. If $\psi_j = \psi_N$ for all $j \geq N$, it is clearly that $\sigma(S_w) = \sigma(\psi_N)$. This completes the proof. $\square$

5. Core Operator

For an invariant subspace M, let R_z and R_w be restrict of T_z and T_w on M respectively. The core operator C for M is defined as

$$C = I - R_zR_z^* - R_wR_w^* + R_zR_wR_z^*R_w^*.$$

The core operator is useful tool in the study of invariant subspace, see [4] for details. Let

$$\Sigma_0 = \|[R_w^*, R_w][R_z^*, R_z]\|_{H.S.}^2, \quad \Sigma_1 = \|[R_z^*, R_w]\|_{H.S.}^2,$$

where $\|\cdot\|_{H.S.}$ means the Hilbert-Schmidt norm. It is well know that $[R_z^*, R_z]$ and $[R_w^*, R_w]$ are orthogonal projections with range $M \ominus zM$ and $M \ominus wM$, respectively, hence $trC^2 = \Sigma_0 + \Sigma_1$. Let $\{g_i\}_{i\geq 0}$ and $\{h_j\}_{j\geq 0}$ be the orthogonal basis of $M \ominus zM$ and $M \ominus wM$, then $\Sigma_0 = \sum_{i,j=0}^{\infty} |\langle g_i, h_j\rangle|^2$.

It is indicated in [14] that if $\mathbb{D}$ is not contained in $\sigma(S_z)$ or $\sigma(S_w)$, both Σ_0 and Σ_1 are finite, moreover, $\Sigma_0 = \Sigma_1 + 1$ (cf. [15]).

For the two inner sequences based invariant subspace M defined in Lemma 2.1, from Theorem 4.3, we know that $\sigma(S_z)\cap\mathbb{D}$ is discrete, so $\Sigma_0 < \infty$ and $\Sigma_1 < \infty$. In the following, we will compute Σ_0. From Lemma 2.1, we have that

$$M \ominus zM = \sum_{j=0}^{\infty} \mathbb{C}\varphi_j(z) \otimes (\psi_j(w)H^2(w) \ominus \psi_{j+1}(w)H^2(w));$$

and

$$M \ominus wM = \sum_{j=0}^{\infty} (\varphi_j(z)H^2(z) \ominus \varphi_{j-1}(z)H^2(z)) \otimes \mathbb{C}\psi_j(w).$$

Let $\{e_{i0}, \ldots, e_{il_i}\}$ be the orthonormal basis of $H^2(z) \ominus \frac{\varphi_{i-1}}{\varphi_i}(z)H^2(z)$ and $\{f_{j0}, \ldots, f_{jd_j}\}$ be the orthonormal basis of $H^2(w) \ominus \frac{\psi_{j+1}}{\psi_j}(w)H^2(w)$. Then $\{\varphi_j\psi_j f_{j0}, \ldots, \varphi_j\psi_j f_{jd_j}\}_{j\geq 0}$ and $\{\varphi_i\psi_i e_{i0}, \ldots, \varphi_i\psi_i e_{il_i}\}_{i\geq 0}$ are the orthonormal basis of $M\ominus zM$ and $M\ominus wM$, respectively. Using the reproducing kernel and some computations, we have

$$\sum_{\beta=0}^{d_j} |f_{j\beta}(0)|^2 = 1 - \left|\frac{\psi_{j+1}}{\psi_j}(0)\right|^2, \quad \sum_{\alpha=0}^{l_i} |e_{i\alpha}(0)|^2 = 1 - \left|\frac{\varphi_{i-1}}{\varphi_i}(0)\right|^2. \quad (5.1)$$

Note that $\{\varphi_j(z)\}_{j\geq 0}$ is a decreasing inner sequence and $\{\psi_j(w)\}_{j\geq 0}$ is increasing, we have

$$\begin{aligned}
\Sigma_0 &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\beta=0}^{d_j}\sum_{\alpha=0}^{l_i} |\langle \varphi_j\psi_j f_{j\beta}, \varphi_i\psi_i e_{i\alpha}\rangle|^2 \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left(\sum_{\beta=0}^{d_j} |\langle \psi_j f_{j\beta}, \psi_i\rangle|^2 \sum_{\alpha=0}^{l_i} |\langle \varphi_j, \varphi_i e_{i\alpha}\rangle|^2\right) \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left(\sum_{\beta=0}^{d_j} \left|\frac{\psi_j}{\psi_i}(0)\right|^2 |f_{j\beta}(0)|^2 \sum_{\alpha=0}^{l_i} |\frac{\varphi_i}{\varphi_j}(0)|^2 |e_{i\alpha}(0)|^2\right)
\end{aligned}$$

$$= \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{\psi_j}{\psi_i}(0)\right|^2\left(1-\left|\frac{\psi_{j+1}}{\psi_j}(0)\right|^2\right)$$
$$\left|\frac{\varphi_i}{\varphi_j}(0)\right|^2\left(1-\left|\frac{\varphi_{i-1}}{\varphi_i}(0)\right|^2\right).$$

So we have the following proposition.

Proposition 5.1. *Let M be the two inner sequences based invariant subspace defined by* (2.1)*, then*

$$trC^2=2\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left(\left|\frac{\varphi_i}{\varphi_j}(0)\right|^2-\left|\frac{\varphi_{i-1}}{\varphi_j}(0)\right|^2\right)\left(\left|\frac{\psi_j}{\psi_i}(0)\right|^2-\left|\frac{\psi_{j+1}}{\psi_i}(0)\right|^2\right)-1. \quad (5.2)$$

Remark 5.2. Given a decreasing inner sequence $\{\varphi_j\}_{j\geq 0}$ and an increasing inner sequence $\{\psi_j\}_{j\geq 0}$, in general, it is hard to determine the quantity (5.2), but from the above computation, we know it is finite!

Remark 5.3. If $\psi_j(w) = w^j$, then $M = \sum_{j=0}^{\infty}\varphi_j(z)H^2(z)w^j$. Then (5.2) becomes $trC^2 = 2\sum_{j=0}^{\infty}\left(1-|\frac{\varphi_{j-1}}{\varphi_j}(0)|^2\right)-1$, where $\varphi_{-1}=0$. This has been obtained in [13].

For a bounded linear operator T, $rankT$ denotes the dimension of the range of T, denoted by $ranT$. In the following, we will study the rank of the core operator. We will focus on when $\Theta(z) = \sum_{j=0}^{K-1}\varphi_j(z)P_j$ is a finite sum, where $\frac{\varphi_j}{\varphi_{j+1}}$ is a nonconstant inner function, $j = 0, 1, \ldots, k-2$, and $\sum_{j=0}^{K-1}P_j = I$. Then M is of the form

$$M = \sum_{j=0}^{K-1}\varphi_j(z)H^2(z)\otimes(\psi_j(w)H^2(w)\ominus\psi_{j+1}(w)H^2(w))$$
$$= \sum_{j=0}^{K-1}(\varphi_j(z)H^2(z)\ominus\varphi_{j-1}(z)H^2(z))\otimes\psi_j(w)H^2(w), \quad (5.3)$$

where $\frac{\psi_{j+1}}{\psi_j}$ is a nonconstant inner function, $j = 0, 1, \ldots, k-2$, and we also use the natation $\varphi_{-1} = 0$, $\psi_0 = 1$ and $\psi_K = 0$. Then by (5.3), it is easy to see that

$$M\ominus zM = \sum_{j=0}^{K-1}\mathbb{C}\varphi_j(z)\otimes(\psi_j(w)H^2(w)\ominus\psi_{j+1}(w)H^2(w)); \quad (5.4)$$

and

$$M\ominus wM = \sum_{j=0}^{K-1}(\varphi_j(z)H^2(z)\ominus\varphi_{j-1}(z)H^2(z))\otimes\mathbb{C}\psi_j(w). \quad (5.5)$$

Note that $\{e_{i0},\ldots,e_{il_i}\}$ is the orthonormal basis of $H^2(z)\ominus\frac{\varphi_{i-1}}{\varphi_i}(z)H^2(z)$ and $\{\varphi_i\psi_ie_{i0},\ldots,\varphi_i\psi_ie_{il_i}\}$ is the orthonormal basis of $(\varphi_iH^2(z)\ominus\varphi_{i-1}H^2(z))$

$\otimes\ \mathbb{C}\psi_i$. For any $f = \sum_{j=0}^{K-1} \varphi_j\psi_j f_j \in M \ominus zM$, where $f_j \in H^2(w) \ominus \frac{\psi_{j+1}}{\psi_j} H^2(w)$, we have

$$\begin{aligned}
[R_w^*, R_w]f &= \sum_{i=0}^{K-1} \sum_{\alpha=0}^{l_i} \langle f, \varphi_i\psi_i e_{i\alpha}\rangle \varphi_i\psi_i e_{i\alpha} \\
&= \sum_{i=0}^{K-1} \sum_{\alpha=0}^{l_i} \left\langle \sum_{j=0}^{K-1} \varphi_j\psi_j f_j, \varphi_i\psi_i e_{i\alpha} \right\rangle \varphi_i\psi_i e_{i\alpha} \\
&= \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} \sum_{\alpha=0}^{l_i} \langle \varphi_j\psi_j f_j, \varphi_i\psi_i e_{i\alpha}\rangle \varphi_i\psi_i e_{i\alpha} \\
&= \sum_{j=0}^{K-1} \sum_{i=0}^{j} \sum_{\alpha=0}^{l_i} \langle \varphi_j\psi_j f_j, \varphi_i\psi_i e_{i\alpha}\rangle \varphi_i\psi_i e_{i\alpha} \\
&= \sum_{j=0}^{K-1} \sum_{i=0}^{j} \sum_{\alpha=0}^{l_i} \frac{\psi_j}{\psi_i}(0)\overline{\frac{\varphi_i}{\varphi_j}(0)} f_j(0)\overline{e_{i\alpha}(0)}\varphi_i\psi_i e_{i\alpha}, \qquad (5.6)
\end{aligned}$$

the fourth equality holds because for $i > j, \langle \varphi_j\psi_j f_j, \varphi_i\psi_i e_{i\alpha}\rangle = 0$. For simplicity, let $\gamma_{ij} = \frac{\psi_j}{\psi_i}(0)\overline{\frac{\varphi_i}{\varphi_j}(0)}$ for $i \leq j$. Set

$$e_i = \sum_{\alpha=0}^{l_i} \overline{e_{i\alpha}(0)}\varphi_i\psi_i e_{i\alpha} \qquad (5.7)$$

It is clear that $e_i \in (\varphi_i H^2(z) \ominus \varphi_{i-1} H^2(z)) \otimes \mathbb{C}\psi_i$. Then

$$\begin{aligned}
[R_w^*, R_w]f &= \sum_{j=0}^{K-1} \sum_{i=0}^{j} \gamma_{ij} f_j(0) e_i \\
&= \sum_{i=0}^{K-1} \sum_{j=i}^{K-1} \gamma_{ij} f_j(0) e_i \\
&= \sum_{i=0}^{K-1} \gamma_i(f) e_i, \qquad (5.8)
\end{aligned}$$

where $\gamma_i(f) = \sum_{j=i}^{K-1} \gamma_{ij} f_j(0)$ is a complex number depending on f. Therefore,

$$rank([R_w^*, R_w][R_z^*, R_z]) \leq K. \qquad (5.9)$$

Lemma 5.4. *For any i=0,1,...,K-1, e_i is defined as (5.7), then $e_i \not\equiv 0$.*

Proof. By (5.1), $\sum_{\alpha=0}^{l_i} |e_{i\alpha}(0)|^2 = 1 - |\frac{\varphi_{i-1}}{\varphi_i}(0)|^2$. Since $\frac{\varphi_{i-1}}{\varphi_i}$ is not a constant, $\sum_{\alpha=0}^{l_i} |e_{i\alpha}(0)|^2 \neq 0$, so $\sum_{\alpha=0}^{l_i} \overline{e_{i\alpha}(0)} e_{i\alpha} \not\equiv 0$. Therefore, $e_i = \sum_{\alpha=0}^{l_i} \overline{e_{i\alpha}(0)}\varphi_i\psi_i e_{i\alpha} = \varphi_i\psi_i \sum_{\alpha=0}^{l_i} \overline{e_{i\alpha}(0)} e_{i\alpha} \not\equiv 0$. □

Theorem 5.5. *Let M be the two inner sequences based invariant subspace defined by (5.3), then $rank C = 2K - 1$.*

Proof. Firstly, we claim that for each $j = 0, 1, \ldots K-2$, there exists $f_j \in H^2(w) \ominus \frac{\psi_{j+1}}{\psi_j} H^2(w)$ such that $f_j(0) \neq 0$, otherwise, $H^2(w) \ominus \frac{\psi_{j+1}}{\psi_j} H^2(w) \subset wH^2(w)$, this implies that $H^2(w) = \frac{\psi_{j+1}}{\psi_j} H^2(w) \oplus wH^2(w)$ is the direct sum of two reducing subspaces, which is a contradiction. It is trivial that there exists $f_{K-1} \in H^2(w)$ such that $f_{K-1}(0) \neq 0$.

We shall show by induction that for any $i = 0, 1, \ldots, K-1$, $e_i \in ran([R_w^*, R_w][R_z^*, R_z])$.

Choose $f_0 \in H^2(w) \ominus \frac{\psi_1}{\psi_0} H^2(w)$ with $f_0(0) \neq 0$. By (5.8),

$$[R_w^*, R_w](\varphi_0 \psi_0 f_0) = \gamma_0(f_0) e_0 = \gamma_{00} f_0(0) e_0. \tag{5.10}$$

Since $\gamma_{00} = 1$ and thus (5.10) implies that $e_0 \in ran([R_w^*, R_w][R_z^*, R_z])$. Now assume that we have shown that $e_0, \ldots, e_{i-1} \in ran([R_w^*, R_w][R_z^*, R_z])$. Choose $f_i \in H^2(w) \ominus \frac{\psi_{i+1}}{\psi_i} H^2(w)$ such that $f_i(0) \neq 0$. Then

$$[R_w^*, R_w](\varphi_i \psi_i f_i) = \sum_{k=0}^{i} \gamma_k(f_i) e_k.$$

Since $\sum_{k=0}^{i-1} \gamma_k(f_i) e_k \in ran([R_w^*, R_w][R_z^*, R_z])$ and $\gamma_i(f_i) = \gamma_{ii} f_i(0) = f_i(0) \neq 0$, hence $e_i \in ran([R_w^*, R_w][R_z^*, R_z])$. This proves the claim and hence $rank([R_w^*, R_w][R_z^*, R_z]) = K$.

By Proposition 2.6 and Theorem 2.7 in [16], $rankC = 2rank([R_w^*, R_w][R_z^*, R_z]) - 1 = 2K - 1$, which completes the proof. $\square$

Example. Let $K = 2$ and $M = \varphi_0(z) H^2(z) \otimes (H^2(w) \ominus \psi_1(w) H^2(w)) \bigoplus H^2(z) \otimes \psi_1(w) H^2(w)$. By Theorem 5.5, $rankC = 3$. In fact, $M = \varphi_0(z) H^2(\mathbb{D}) + \psi_1(w) H^2(\mathbb{D})$. This type of invariant subspace has been studied in [17] and $rankC$ was also obtained by a different method.

Remark 5.6. If the two inner sequences based invariant subspace M is defined by nontrivial infinite sum Θ, then C is of infinite rank.

Acknowledgements

The author would like to thank Professor Rongwei Yang in the SUNY at Albany for his valuable suggestions and supervision. The author also gratefully acknowledge the referee, whose advices and comments considerably improved the original version of this paper.

References

[1] Agrawal, O.P., Clark, D.N., Douglas, R.G.: Invariant subspaces in the polydisk. Pac. J. Math. **121**, 369–373 (1986)

[2] Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math. **81**, 239–255 (1948)

[3] Douglas, R.G., Paulsen, V.I.: Hilbert modules over function algebras. Longman Research Notes in Mathematics, Vol. 217. Longman, Harlow (1989)

[4] Guo, K., Yang, R.: The core function of submodules over the bidisk. Indiana Univ. Math. J. **53**, 205–222 (2004)

[5] Halmos, P.R.: Shifts on Hilbert space. J. Reine Angew. Math. **208**, 102–112 (1961)

[6] Izuchi, K.J., Izuchi, K.H., Izuchi, Y.: Ranks of invariant subspaces of the Hardy space over the bidisk. J. Reine Angew. Math. **659**, 101–139 (2011)

[7] Izuchi, K.J., Izuchi, K.H., Izuchi, Y.: Blaschke products and the rank of backward shift invariant subspaces over the bidisk. J. Funct. Anal. **261**, 1457–1468 (2011)

[8] Izuchi, K., Nakazi, T., Seto, M.: Backward shift invariant subspaces in the bidisc III. Acta Sci. Math. (Szeged) **70**(3-4), 727–749 (2004)

[9] Lax, P.: Translation invariant spaces. Acta Math. **101**, 163–178 (1959)

[10] Nagy, Sz., Foias, C.: Harmonic Analysis of Operators on Hilbert Space. North Holland, Amsterdam (1970)

[11] Qin, Y., Yang, R.: A characterization of submodules via Beurling–Lax–Halmos theorem, Proc. Am. Math. Soc (2013, to appear)

[12] Seto, M.: Infinite sequences of inner functions and submodules in $H^2(\mathbb{D}^2)$. J. Oper. Theory **61**, 75–786 (2009)

[13] Seto, M., Yang, R.: Inner sequence based invariant subspaces in $H^2(D^2)$. Proc. Am. Math. Soc. **135**, 2519–2526 (2007)

[14] Yang, R.: Operator theory in the Hardy space over the bidisk III. J. Funct. Anal. **186**, 521–545 (2001)

[15] Yang, R.: Beurling's phenomenon in two variables. Inter. Equ. Oper. Theory **48**, 411–423 (2004)

[16] Yang, R.: The core operator and congruent submodules. J. Funct. Anal. **228**, 521–545 (2005)

[17] Yang, R.: Hilbert–Schmidt submodules and issues of unitary equivalence. J. Oper. Theory. **53**, 169–184 (2005)

Yixin Yang (✉)
Department of Mathematics and Statistics
SUNY At Albany
Albany, NY 12222
USA
e-mail: yy281612@albany.edu

Received: March 18, 2013.
Revised: July 22, 2013.